A 4-DIMENSIONAL GRAPH HAS AT LEAST 9 EDGES

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ABSTRACT. The open problem posed by Paul Erdős asking for the smallest number of edges in a 4-dimensional graph is solved by showing that a 4-dimensional graph must have at least 9 edges. Furthermore, there is only one 4-dimensional graph with 9 edges, namely $K_{3,3}$.

1. INTRODUCTION

In [1, p88] and [2] the dimension of a graph is defined as follows.

Definition 1. The *dimension* of a graph G, denoted $dim(G)$, is the minimum n such that G has a unit-distance representation in $\mathbb{R}^n$, i.e., every edge is of length 1. The vertices of G are mapped to distinct points of $\mathbb{R}^n$, but edges may cross.

For example, a path has dimension 1, a cyclic graph has dimension 2, and K_4 has dimension 3 (because it can be embedded in 3-space as a regular tetrahedron with edge length 1). A somewhat less obvious case is shown in Figure 1 where graph G151 is shown as it appears in [3, p10] and with a unit-distance representation (see Figure 9 for a correspondence between the vertices of the two representations).

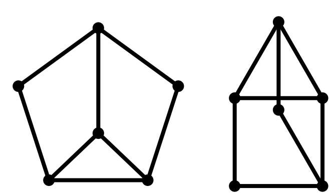

FIGURE 1. Graph G151 and a unit-distance representation in the plane.

In [1, p93] a problem posed by Paul Erdős in 1991 is stated as follows.

Problem 2. What is the smallest number of edges in a graph G such that $dim(G) = 4$?

The answer to this question is 9, as is shown in the remainder of this article. It is also shown that there is only one 4-dimensional graph with 9 edges, namely $K_{3,3}$.

2. BASIC FACTS ABOUT GRAPH DIMENSION

Basic results on the dimensions of graphs are given in [1, pp88-93] and [2]. Here are results used in this article.

Proposition 3. *Basic results on the dimension of a graph are as follows:*
$dim(K_n) = n - 1$
$dim(K_n - e) = n - 2$
$dim(K_{1,1}) = dim(K_{1,2}) = 1$, $dim(K_{1,m}) = 2$ *for* $m \geqq 3$
$dim(K_{2,2}) = 2$, $dim(K_{2,m}) = 3$ *for* $m \geqq 3$
$dim(K_{m,n}) = 4$ *for* $m \geqq n \geqq 3$
$dim(C_n) = 2$ *for* C_n *a cyclic graph of order* $n \geqq 3$
$dim(tree) \leqq 2$
if H *is a subgraph of* G *then* $dim(H) \leqq dim(G)$

Since $K_{2,3}$ plays an important role in some arguments below, and to give a flavor of the kind of proofs found in [1] and [2], we give a proof that $dim(K_{2,3}) = 3$.

Proposition 4. *If two vertices* u *and* v *of a graph* G *have at least three neighbors in common, then* $dim(G) \geqq 3$.

Proof. Consider two circles in the plane of radius 1 with u and v as centers. Then the three neighbors in common must all lie on both circles since each of the neighbors is at distance 1 from both u and v. But two circles can intersect in at most 2 points. So a unit-distance representation in the plane is impossible. Thus $dim(G) \geqq 3$. □

Corollary 5. $dim(K_{2,3}) = 3$.

Proof. Let u and v be the two vertices of degree 3. Then u and v have 3 neighbors in common, so by the proposition, $dim(G) \geqq 3$. But, $K_{2,3}$ can be embedded in 3-space so that all edges are unit length as shown in Figure 2. Place u on the positive z-axis at $(0, 0, d)$, where $d = \sqrt{2}/2$, and place v on the negative z-axis at $(0, 0, -d)$. Place the three vertices of degree two in the xy-plane at $(d, 0, 0), (-d, 0, 0)$, and $(0, d, 0)$. The distances from the latter three points to the points on the z-axis are all $\sqrt{d^2 + d^2} = 1$. □

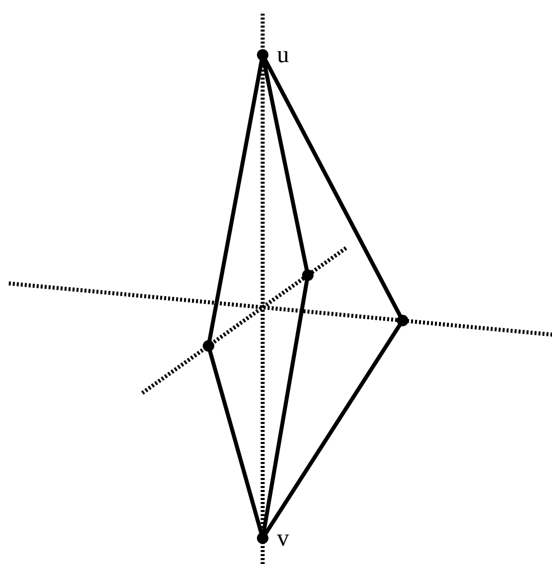

FIGURE 2. $K_{2,3}$ embedded in 3-space with all edges unit length.

Many other embeddings are possible. For example, if u and v are on the z-axis at distance h from the origin, $0 < h < 1$, then the other vertices can be anywhere on a circle in the xy-plane with radius $\sqrt{1-h^2}$ and center at the origin.

Note that $K_{2,3}$ is a counterexample to the conjecture that planar graphs have dimension 2. $K_{2,3}$ is planar (see Figure 3), but it has dimension 3. The existence of a planar representation of a graph does not imply that a planar unit-distance representation exists.

3. 3-ROUTES

We introduce a type of graph which will play a role in restricting the set of graphs we need to examine to prove the main result.

Definition 6. A *3-route* is a graph with four or more vertices, two of which, u, v, $u \neq v$, are distinguished. There are exactly three paths from u to v, disjoint except for the endpoints u and v. A 3-route with path lengths i, j, k is denoted $R_{i,j,k}$.

Example 7. The minimal 3-route is $R_{2,1,2}$ as shown in Figure 3. The minimal 3-route with all paths containing more than one edge is $R_{2,2,2}$. Note that $R_{2,2,2} \cong K_{2,3}$.

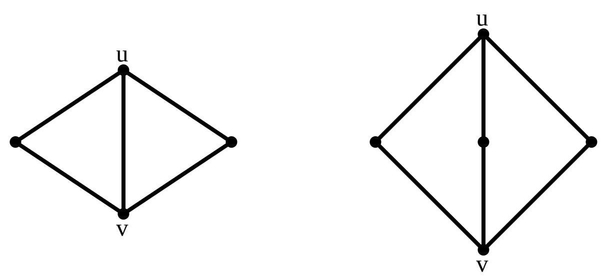

FIGURE 3. Minimal 3-routes $R_{2,1,2}$ and $R_{2,2,2} \cong K_{2,3}$.

We would like to determine the dimension of a 3-route $R_{i,j,k}$. Without loss of generality we can assume that $j \leqq i \leqq k$, i.e., the middle path is the shortest and the right path is the longest. Note that there can be at most one path of length 1 (otherwise there would be multiple edges from u to v), so both the left and right paths must have lengths greater than 1. Figure 4 shows the first six left paths, Figure 5 shows the first five middle paths, and figure 6 shows the first six right paths.

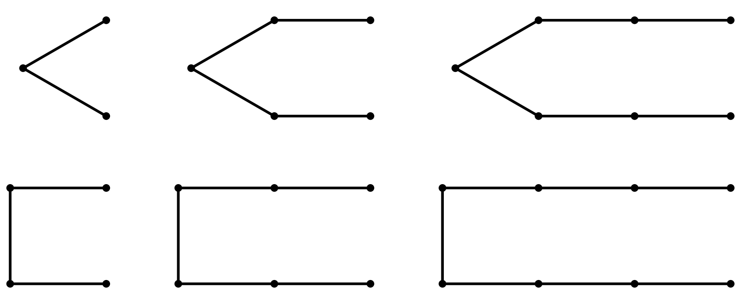

FIGURE 4. Embeddings of left paths.

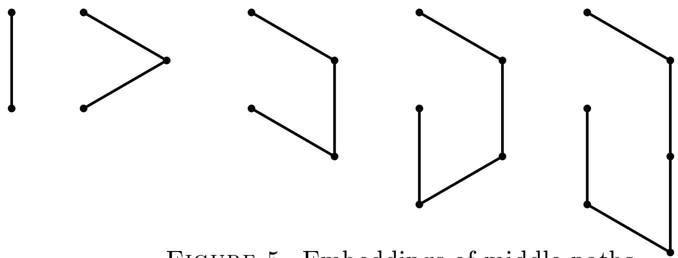

FIGURE 5. Embeddings of middle paths.

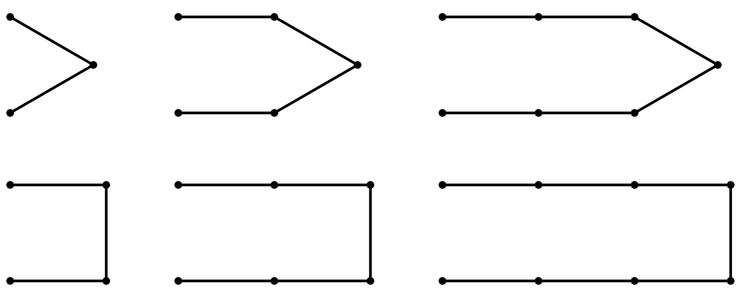

FIGURE 6. Embeddings of right paths.

To create a unit-distance representation in the plane of any $R_{i,j,k}$ with $j \leqq i \leqq k$, simply create a left path of length i by following the pattern in Figure 4, create a middle path of length j by following the pattern in Figure 5, and create a right path of length k by following the pattern in Figure 6. Figure 7 shows examples of such embeddings.

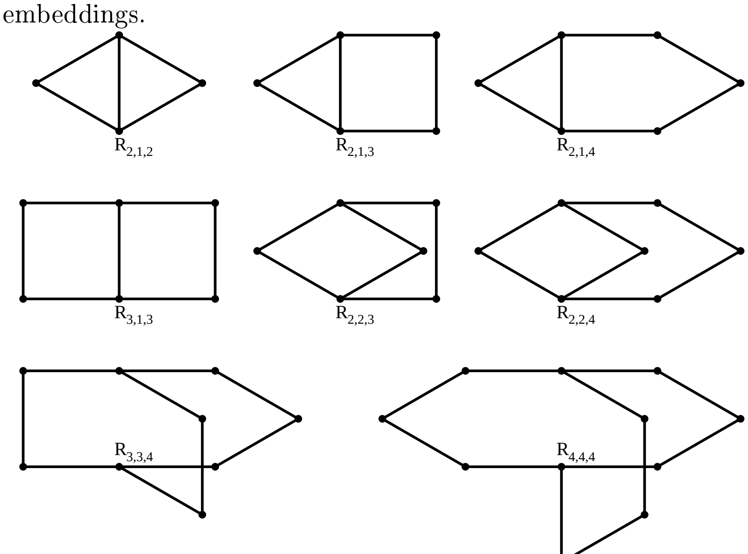

FIGURE 7. Examples of unit-distance representations of 3-routes.

We must make sure that this construction does not result in two coincident vertices. Clearly a left path and a right path never conflict, nor do a left path and a middle path. The only possible conflict might occur between a middle path and a right path, and, indeed, this happens when both are of length 2. Note that since the middle path is the shortest and the right path is the longest, this means the left path is also of length 2, so the graph is $\mathrm{R}_{2,2,2}$ which is isomorphic to $K_{2,3}$, a 3-dimensional graph. Thus we have established the following:

Proposition 8. *Every 3-route is 2-dimensional except $R_{2,2,2}$ which is 3-dimensional.*

It is well-known that if an edge is added to a tree, the resulting graph contains exactly one cycle. What if two edges are added to a tree? The first edge creates a cycle. The second edge also creates a cycle, which is related to the first cycle in one of two ways: Either the two cycles have no edge in common, or the two cycles share one or more edges. In the former case, the graph is clearly 2-dimensional. In the latter case, the two cycles form a 3-route, so the graph is 2-dimensional unless it is $R_{2,2,2}$. In all cases, a tree with two edges added to it has dimension $\leqq 3$.

4. FINDING ALL CANDIDATE GRAPHS

Only biconnected graphs need be considered, as is shown by the following proposition.

Proposition 9. *Let G be a connected graph of dimension 4 having a minimal number of edges. Then G is biconnected.*

Proof. Assume G is not biconnected and let v be a cut vertex. Let B be any block in the decomposition of G determined by v. Since B has fewer edges than G and G is a 4-dimensional graph with a minimum number of edges, it must be that $dim(B) \leqq 3$. Thus all blocks can be represented in the plane or in 3-space with unit length edges, and then joined at v in such a way that all vertices of G are distinct points. Thus $dim(G) \leqq 3$, a contradiction showing that G is biconnected. □

We want to find one or more graphs of dimension 4 with the least number of edges. Since $dim\,(K_5) = 4$ and $dim\,(K_5 - e) = 3$, it is clear that we need not consider graphs with 5 or fewer vertices because other than K_5 each of them is isomorphic to a subgraph of $K_5 - e$ and hence has dimension $\leqq 3$.

Turning to graphs of order 6, we immediately have a better candidate than K_5, namely, $K_{3,3}$, which has dimension 4 and 9 edges. Therefore, we need not consider any graphs with more than 9 edges. A graph with 9 edges can have at most 10 vertices, in which case it is a tree. But a tree has dimension $\leqq 2$, so we need only consider graphs of orders 6, 7, 8, and 9.

In Table 1 we see that we need only consider graphs with 7 vertices and 9 edges and graphs with 6 vertices and 8 or 9 edges. To justify this observation: If a graph has n edges, then the maximum number of vertices it can have is $n+1$. In this case, the graph is a tree, so its dimension is $\leqq 2$. If one more edge is added to a tree, then the resulting graph has exactly one cycle. Clearly this graph has dimension 2. If yet another edge is added, the graph has at least two cycles. If the two cycles have no edge in common, then the dimension is 2. If the two cycles do have at least one edge in common, then the graph is a 3-route, and, hence, by Proposition 8 the dimension is $\leqq 3$. Thus, all combinations of orders and sizes shown in Table 1 are ruled out except for graphs with 7 vertices and 9 edges and graphs with 6 vertices and 8 or 9 edges.

Consulting [3, pp34-35, 40-42] we see the number of non-isomorphic, biconnected (vertex-connectivity$\geqq 2$) graphs in each of the three categories we need to consider: A total of $14 + 9 + 20 = 43$ graphs.

TABLE 1. Orders and sizes of candidate graphs.

no. edges no. verts	9	8	7	6	5
6	14	9	3-route	cycle	tree
7	20	3-route	cycle	tree	
8	3-route	cycle	tree		
9	cycle	tree			
10	tree				

5. THE SOLUTION

Of the 43 candidate graphs, 27 are 2-dimensional and 15 are 3-dimensional. For most of the 2-dimensional graphs shown in Figure 8, it is obvious that they are isomorphic to the diagrams found in [3, pp10-11, 18]. There are, however, three 2-dimensional graphs for which the unit-distance representation is not immediately obvious: G151, G573, and G580. Figure 9 shows these graphs, each with a pair of diagrams. The graph names have suffixes a and b, where a is a diagram from [3], and b is an representation from Figure 8. The vertices are labeled with letters so that it is easy to determine that the graph in an a diagram is isomorphic to the graph in the corresponding b diagram.

The diagrams for G151 and G580 make the representations in the plane rather evident. For graph G573, however, it is not so clear that in diagram G573b all edges are of unit length. This can be verified from the coordinates of the vertices of G573b shown in Table 2

TABLE 2. Coordinates of the vertices of G573b.

	x	y
A	$\sqrt{37}/8$	$3\sqrt{3}/8$
B	$\sqrt{37}/8 - 9\sqrt{301}/344 + 1/2$	$-\sqrt{903}/86$
C	$1 + \sqrt{37}/8$	$-3\sqrt{3}/8$
D	1	0
E	0	0
F	$\sqrt{37}/8$	$-3\sqrt{3}/8$
G	$1 + \sqrt{37}/8$	$3\sqrt{3}/8$

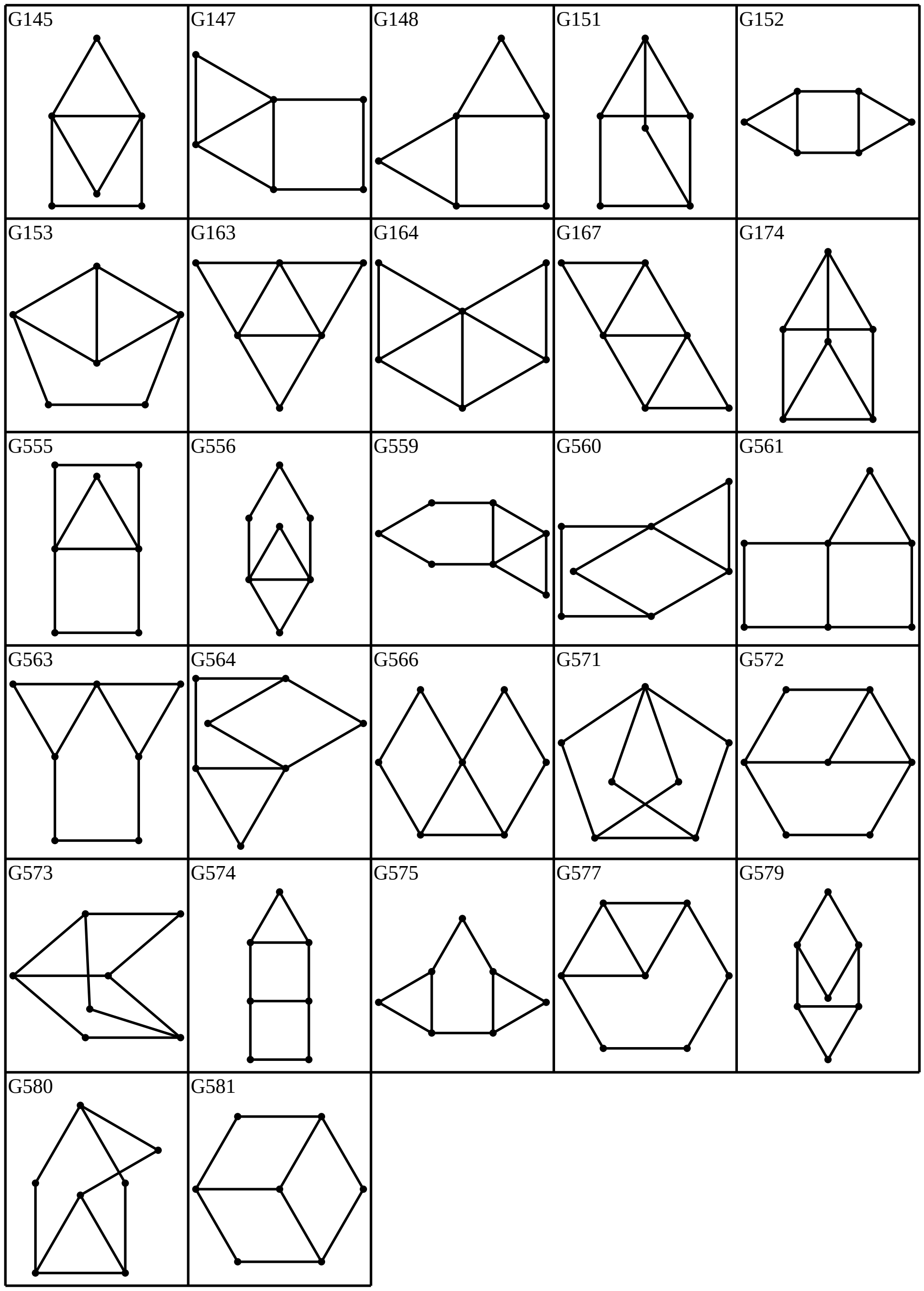

FIGURE 8. Unit-distance representations of 2-dimensional graphs in the plane.

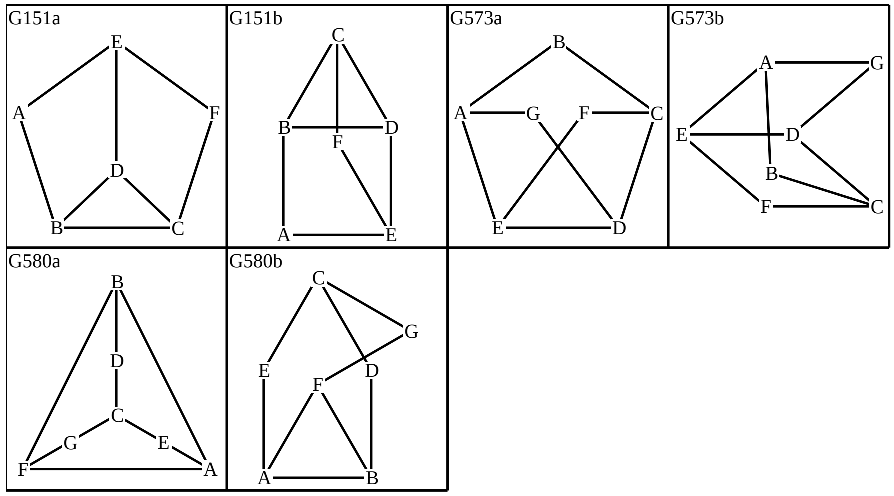

FIGURE 9. 2-dimensional unit-distance representations compared to *An Atlas of Graphs*.

The 15 3-dimensional graphs are shown in Figures 11 and 12. For all these graphs except two, $K_{2,3}$ is a subgraph, which guarantees that the dimension is at least 3. In all cases, the $K_{2,3}$ subgraph is embedded with A on the positive z-axis, B on the negative z-axis, and C, D, and E on the x- and y-axes. In most cases, A and B are at distance $\sqrt{3}/2$ from the origin, and C, D, and E are at distance $1/2$ from the origin. (For G161 and G162, A and B are at distance $1/2$ from the origin, and C, D, and E are at distance $\sqrt{3}/2$).

Graph G169 does not have $K_{2,3}$ as a subgraph. However, it does have K_4 as a subgraph, so its dimension must be at least $dim(K_4) = 3$. The diagram for G169 shows the K_4 subgraph as a regular tetrahedron.

Graph G171 does not have $K_{2,3}$ nor K_4 as a subgraph. Nevertheless $dim(G171) = 3$. To see this, assume that $dim(G171) = 2$, so we have a unit-distance representation in the plane. Since G171 contains the 3-cycles AEF, EDF, and DCF, these must appear in the embedding as equilateral triangles with unit edge lengths. There is only one way to embed this, as shown in Figure 10.

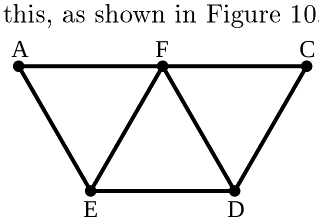

FIGURE 10. Three adjacent 3-cycles in G171.

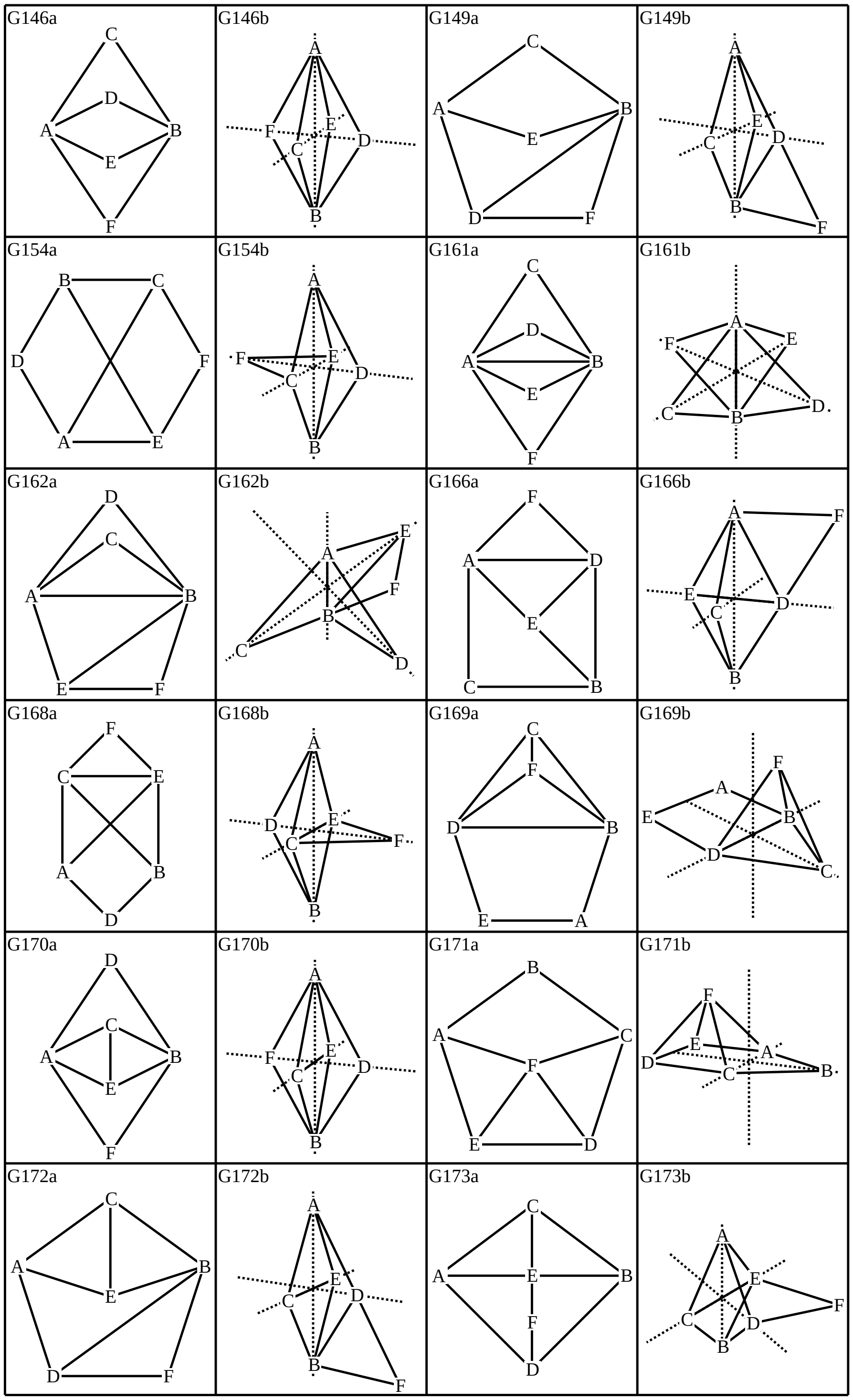

FIGURE 11. Embeddings of 3-dimensional graphs in 3-space - part 1.

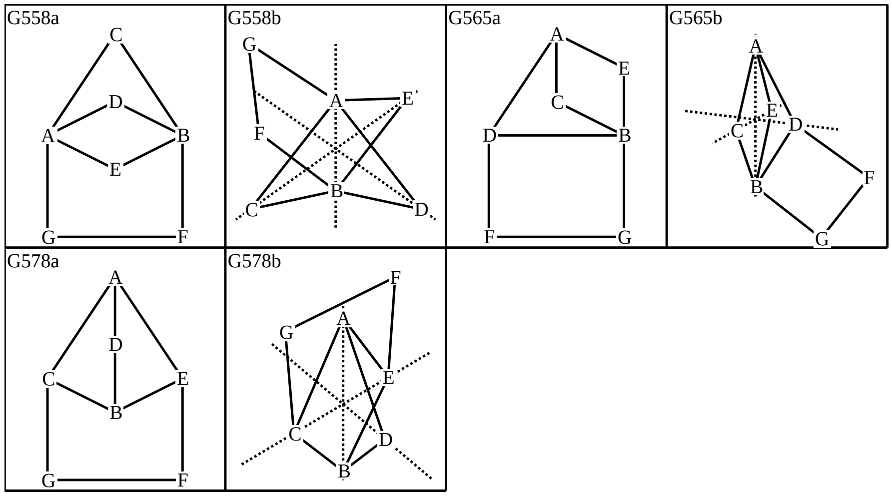

FIGURE 12. Embeddings of 3-dimensional graphs in 3-space - part 2.

Since vertex B must be at unit distance from both A and C, there is only one place to embed it, namely where vertex F is already embedded. So there is no way to embed G171 in the plane with vertices B and F at distinct points. Thus $dim(G171) = 3$.

Of the 43 candidate graphs, 42 have been shown to have dimension 2 or 3. The only remaining graph is $K_{3,3}$, which is 4-dimensional. Thus the minimum number of edges which a 4-dimensional graph can have is 9, and there is only one such graph, namely $K_{3,3}$.

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